Final EXAM : MTH 213, ~~Spring~~ Summer 2018

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Score = $\frac{60}{63}$

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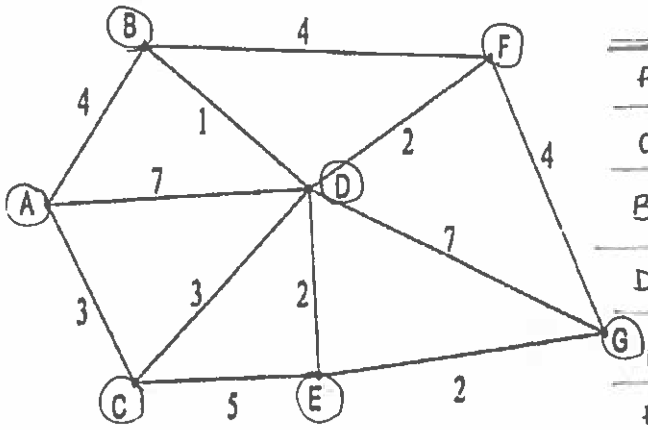
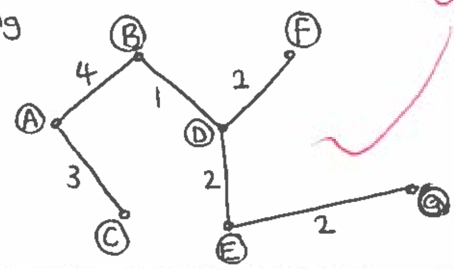
QUESTION 1. (7 points)
Use Dijkstra's method to find the minimum spanning tree of the below graph.

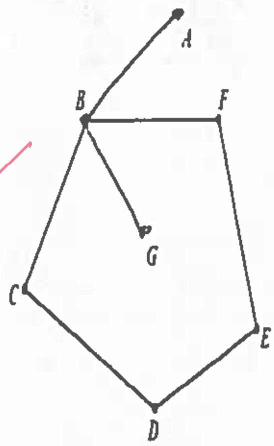


	A	B	C	D	E	F	G
A	$\boxed{0}$	4^A	3^A	7^A	∞	∞	∞
C		∞ 4^A	$\boxed{3^A}$	3+3 6^C	3+5 8^C	∞	∞
B		$\boxed{4^A}$		4+1 5^B	∞ 8^C	4+4 8^B	∞
D				$\boxed{5^B}$	2+5 7^D	2+5 7^D	7+5 12^D
E					$\boxed{7^D}$	∞ 7^D	2+7 9^E
F						$\boxed{7^D}$	4+7 9^E
G							$\boxed{9^E}$

∴ Minimum spanning tree =



QUESTION 2. (3 points)



Stare at the graph. You are allowed to add only EDGES (no vertices) so that the graph becomes connected and an Euler Path. that is not Eulerian
You may show the work on the graph itself.

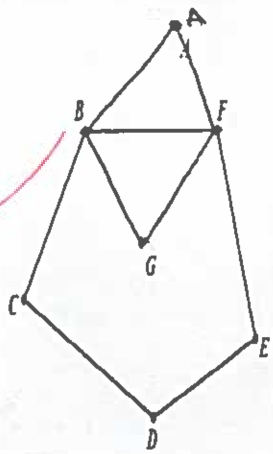
Euler Path → from starting vertex v, can visit each ~~vertex~~ edge exactly once and end at different vertex y.

⇒ Exactly 2 vertices of odd degrees.

[deg(A) = deg(G) = 1 . Rest are even]

[Path example : A – B – F – E – D – C – B – G].

QUESTION 3. (3 points)



Stare at the graph. You are allowed to add only EDGES (no vertices) so that the graph becomes connected and Eulerian (Euler circuit).
You may show the work on the graph itself.

Eulerian → from starting vertex v, can visit each edge exactly once and return to starting vertex.

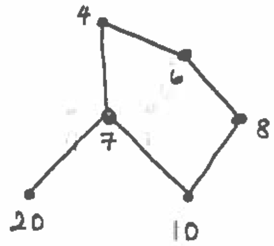
⇒ All vertices must be of even degrees.

[Circuit example : A – B – F – E – D – C – B – G – F – A].

QUESTION 4. $A = \{4, 6, 7, 8, 10, 20\}$. Define $\leq$ on A such that $\forall a, b \in A$, $a \leq b$ if and only if $b - a \in \{0, -2, -3, -4, -6, -13, -16\}$. Then $(A, \leq)$ is a partially ordered set (DO NO SHOW THAT)

(i) **(4 points)** Draw the Hassee diagram of such relation

- 4 "≤" : (4).
- 6 "≤" : 4, (6).
- 7 "≤" : 4, (7).
- 8 "≤" : 4, 6, (8).
- 10 "≤" : 4, 6, ~~7~~, 8, (10).
- 20 "≤" : 4, 7, (20).



(ii) **(3 points)** By staring at the Hassee diagram, if possible, find

a. $20 \vee 10$ = 7.

b. $7 \wedge 6$ = 10.

c. $20 \vee 8$ = 4.

d. $20 \vee 4$ = 4.

e. Is there a $c \in A$ such that $a \leq c$ for every $a \in A$? If yes, find c ⇒ Yes, c = 4.

f. Is there an $m \in A$ such that $m \leq a$ for every $a \in A$? If yes, find m ⇒ No, DNE

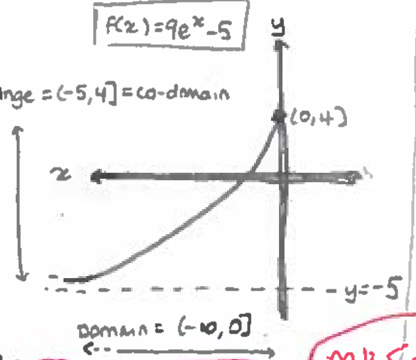
QUESTION 5. (3 points) Convince me that $|(-\infty, 0]| = |(-5, 4]|$ by constructing a bijective function between the two sets.

Let $f : (-\infty, 0] \longrightarrow (-5, 4]$

$\therefore f(x) = 9e^x - 5$.

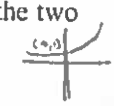
$\boxed{f(x) = 9e^x - 5}$

Range = (-5, 4] = co-domain

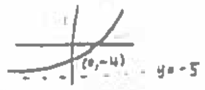


Domain = (-∞, 0]

e^x =



$e^x - 5$ =



$me^0 - 5 = 4$

$m - 5 = 4 \Rightarrow m = 9$.

From graph, it is clear that f(x) is one-to-one (since function is increasing) and onto (since range = co-domain).

Hence f is a bijective function.

We know from fact that, if a bijective function can be built, the domain & codomain have same cardinality.

$\therefore |(-\infty, 0]| = |(-5, 4]|$ QED.

QUESTION 6. (6 points)

(i) Let F be a set with 4 elements, Consider the power set of A, $P(A)$, and let $H = \{d \subset F \mid |d| = 3\}$. Find $|H|$ (i.e., find the cardinality of H)

my fault

irrelevant

$|F| = 4$.

H is a set containing all subsets of F for which cardinality of subset = 3.

$\therefore |H|$ = # of all possible subsets d, where $|d| = 3$. (order in subsets not important)

Hence $|H| = \underline{4C3} = 4$.

6 ~~4~~

my fault, supposed to be P(A), so 16C3

(ii) How many 4-digit even integers greater than 5300 can be formed using the digits (2, 3, 4, 5, 6, 7, 8) such that the third digit must be an even integer.

1^{st} = 4C1 (>5000)

2^{nd} = 6C1 (>300)

3^{rd} = 4C1 (even; {2, 4, 6, 8})

4^{th} = 4C1 (even integer).

∴ Total possibilities = 4C1 × 6C1 × 4C1 × 4C1 = 384.

(-1) 1×6×4×4 + 3×7×4×4

(iii) There are 432 balls and there are 9 holes (very deep holes). The holes are labeled A, A, A, B, B, B, C, C, C. 123 balls must be placed in A-holes (i.e., maybe all of them in one A-hole, or in two A-holes or in three A-holes), 200 balls must be placed in B-holes (see my earlier comment), and the remaining balls must be placed in C-Holes (again, see my earlier comment). Then there are at least n balls that are placed in the same hole (such hole could be an A-hole, or a B-hole, or a C-hole). What is the maximum value of n?

|Domain A| = 123. |Co-domain A| = 3.

|Domain B| = 200. |Co-domain B| = 3.

|Domain C| = 432 − 323 = 109. |Co-domain C| = 3.

⇒ least balls in A = $\lceil \frac{123}{3} \rceil$ = 41.

least balls in B = $\lceil \frac{200}{3} \rceil$ = 67.

least balls in C = $\lceil \frac{109}{3} \rceil$ = 37.

∴ least balls placed in same hole of any kind = min[41, 67, 37]

∴ $\boxed{n = 37}$.

Answer 67

QUESTION 7. (2 points) Find 8^{24002} (*mod* 35)

Fact: If gcd(a, n) = 1, then $a^{\phi(n)} \pmod{n} = 1$.

n = 35 = 5 × 7.

$\phi(n) = 4 \times 6 = 24$.

gcd(8, 35) = 1.

$\therefore 8^{24} \pmod{35} = 1$.

24002 = 24000 + 2

= 1000(24) + 2

$\therefore 8^{24002} \pmod{35} = 8^{24000+2} \pmod{35}$

$= 8^{1000(24)} \cdot 8^2 \pmod{35}$

$= 8^{1000(24)} \pmod{35} \times 8^2 \pmod{35}$

$= 1 \times 64 \pmod{35}$

= 29.

$\therefore 8^{24002} \pmod{35} = 29$.

Summer

QUESTION 8. (3 points)
Given $S = \{1,2,3,4,5,6,7,8\}$. Let f be a bijective function from S onto S such that

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 6 & 5 & 3 & 4 & 2 & 1 & 8 \end{pmatrix}$$

(i) Find f^2 (i.e., find $f o f$). (Note that by staring at f, we understand that $f(1) = 7, \ldots, f(8) = 8$)

$f^2 = f o f = f(f(2))$.

$$\therefore f^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 4 & 5 & 3 & 6 & 7 & 8 \end{pmatrix}$$

(ii) Find the least positive integer n such that $f^n = I$, where I is the identity map (i.e., $I(a) = a$ for every $a \in S$)

f as disjoint cycle $= \underbrace{(1\ 7)}_{\text{2-cycle}}\underbrace{(2\ 6)}_{\text{2-cycle}}\underbrace{(3\ 5\ 4)}_{\text{3-cycle}}$.

$\therefore n = \mathrm{LCM}[2,2,3] = \underline{6}$. $\therefore \underline{f^6 = I}$.

QUESTION 9. (3 points) Let $A = \{0,1,2,3,4\}$ and consider the following equivalence classes of an equivalence relation "=" on A: $[0] = \{0,3,4\}$, $[1] = \{1\}$, $[2] = \{2\}$. Note that the definition of "=" is not given here. Write down the elements (explicitly) of "=" as a subset of $A \times A$.

$[0] = \{0,3,4\}$.
$[1] = \{1\}$
$[2] = \{2\}$.

$\therefore$ Elements of "=" as subset of $A \times A$
$= (0,0), (0,3), (0,4), (3,0), (3,3), (3,4), (4,0), (4,3), (4,4),$
$(1,1), (2,2)$.

QUESTION 10. (4 points) Let $A = \{1,2,4,7,8,11,13,14\}$ and let $H = \{1,4,7,13\}$. Define "=" on A such that $\forall a,b \in A$, $a" = "b$ if and only if $ab\ (mod 15) \in H$. Then "=" is an equivalence relation. Do not show that.

(i) Does $3 = 7$? why? $\Rightarrow$ If $3"="7$, this means $(3)(7)(\text{mod } 15) \in H$. $(3)(7)(\text{mod } 15) = 21(\text{mod } 15) = 6 \notin H$.
Hence, $\boxed{3 "\neq" 7}$.

(ii) Does $11 = 14$? why?
If $11"="14$, then $(11)(14)(\text{mod } 15) \in H$. $(11)(14)(\text{mod } 15) = 154(\text{mod } 15) = 4 \in H$.
Hence $\boxed{11 "=" 14}$.

(iii) Find all equivalence classes of A

- $[1] = \{1,4,7,13\}$.
- $[2] = \{2,8,11,14\}$.

QUESTION 11. (4 points) Let $m = gcd(34, 126)$. Then find a, b such that $m = 34a + 126b$

$126 = 34 \times 3 + 24 \Rightarrow 34 = 24 \times 1 + 10 \Rightarrow 24 = 10 \times 2 + 4 \Rightarrow 10 = 4 \times 2 + 2 \Rightarrow 4 = 2 \times 2 + 0$ (stop)

$\therefore \underline{gcd(34,126)} = 2$.

$2 = 10 - 8$
$= 10 - (4 \times 2)$
$= 10 - ((24-20) \times 2)$
$= 10 - (24 \times 2) + (20 \times 2)$
$= (10 \times 1) - (24 \times 2) + (10 \times 4)$
$= (10 \times 5) - (24 \times 2)$
$= ((34-24) \times 5) - (24 \times 2)$
$= (34 \times 5) - (24 \times 5) - (24 \times 2)$
$= (34 \times 5) - (24 \times 7)$
$= (34 \times 5) - ((126-102) \times 7)$
$= (34 \times 5) - (126 \times 7) + (102 \times 7)$
$= (34 \times 5) - (126 \times 7) + ((34 \times 3) \times 7)$
$= (34 \times 5) - (126 \times 7) + (34 \times 21)$
$= (34 \times 26) + (126 \times -7)$

$\therefore \boxed{m = 2.\ a = 26.\ b = -7}$

QUESTION 12. (4 points) Use math induction and convince me that $5^{2m}\ (mod\ 20) = 5$ for every $m \geq 1$.

① Prove it for smallest possible m ie. m=1 :

when $m = 1$:

$5^{2m}(\text{mod } 20) = 5^{2(1)}(\text{mod } 20)$
$= 5^2(\text{mod } 20)$
$= 25\ (\text{mod } 20) = 5$.

② Assume claim is valid for some $m=n$ where $n>1$:

ie. assume $5^{2n}(\text{mod } 20) = 5$.

③ Prove it for (n+1) :

$5^{2(n+1)}(\text{mod } 20) = 5^{2n+2}(\text{mod } 20)$
$= 5^{2n} \cdot 5^{2}\ (\text{mod } 20)$
$= 5^{2n}(\text{mod } 20) \times 5^2(\text{mod } 20)$.
$\Downarrow$ ② $\qquad \Downarrow$ ①
$= (5 \times 5)(\text{mod } 20)$
$= 25\ (\text{mod } 20) = 5$.

Hence $5^{2m}(\text{mod } 20) = 5 \ \forall\, m \geq 1$.

QUESTION 13. (4 points) Let X be number of students in MTH 213. Given X lives in PLANET Z_{70} such that $X\ (mod\ 7) = 2$ and $2X\ (mod 10) = 6$. Use the Chinese remainder Theorem (CRT) and find all possible values of X.

X lives in planet $Z_{70} \Rightarrow 0 \leq X < 70$.

Given : $X(\text{mod } 7) = 2$
$2X(\text{mod } 10) = 6$.

* Solve $2X(\text{mod } 10) = 6$ over Z_{10} :

$2X(\text{mod } 10) = 6$ has 2 solutions over Z_{10} (since $gcd(2,6)=2$ & $2|10$) .

∴ [redacted] $x = 3$ & $x = 8$.

* Check if CRT applies :

$gcd(7,10) = 1$.

Hence, CRT is applicable .

*CRT :

$n = n_1 n_2 = (7)(10) = 70$.

∘ $m_1 = \frac{n}{n_1} = \frac{70}{7} = 10$.

$10x_1(\text{mod } 7) = 1$.

∴ $\boxed{x_1 = 5}$

∘ $m_2 = \frac{n}{n_2} = \frac{70}{10} = 7$.

$7x_2(\text{mod } 10) = 1$.

∴ $\boxed{x_2 = 3}$

we know $0 \leq x < n$

since given eqn has 2X, we have two possible solutions in planet Z_{70}

∘ when $r_2 = 3$

∘ when $r_2 = 8$.

$X = (m_1x_1r_1 + m_2x_2r_2)(\text{mod } n)$.

[redacted]

∘ $X_1 = ((10)(5)(2) + (7)(3)(3))(\text{mod } 70)$
$= 163(\text{mod } 70)$
$= 23$.

∘ $X_2 = ((10)(5)(2) + (7)(3)(8))(\text{mod } 70)$
$= 268(\text{mod } 70)$
$= 58$.

∴ $X = \{23, 58\}$.

QUESTION 14. (6 points) JUST WRITE T OR F

(i) $\exists! x \in Z$ such that $x^2 - 6x + 9 = 0$. $\longrightarrow$ T.

(ii) $\{5, \{5\}\} - \{5\} = \{5\}$ $\longrightarrow$ F (should be $\{\{5\}\}$) .

(iii) $\forall x \in R, \exists y \in Q^*$ such that $xy \in Q$. $\longrightarrow$ F.

(iv) If $x^2 = 2$ for some $x \in N$, then $xy = \pi\ \forall y \in N$. $\longrightarrow$ F.

(v) If $-7(mod\ 10) = 23(mod\ 10)$, then $-3(mod\ 5) = 23\ (mod\ 5)$ $\longrightarrow$ F.

(vi) $\{3, \{3\}\} \in P(A)$, where $A = \{3, \{3\}, \phi\}$. $\longrightarrow$ T.

-1

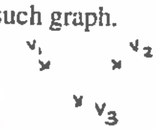
QUESTION 15. (4 points) Is the sequence 3, 2, 2, 1, 1, 1 graphical? If yes draw such graph.

$S = 3, 2, 2, 1, 1, 1$.

∴ $S' = 1, 1, 0, 1, 1$
$= 1, 1, 1, 1, 0$.

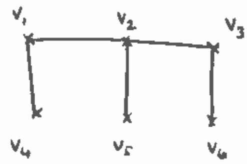
$S'' = 0, 1, 1, 0$
$= 1, 1, 0, 0$.

$S''' = 0, 0, 0$. $\Longrightarrow S''' =$



Since S''' is graphical, this means original sequence S is also graphical .

∴ S =



$deg(v_2) = 3$
$deg(v_1) = 2$
$deg(v_3) = 2$
$deg(v_4) = 1$
$deg(v_5) = 1$
$deg(v_6) = 1$.

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